Function Theory of a Complex Variable (E2): Exercise sheet 2 Solutions

1. Writing $f = u + iv$, $z = x + iy$, we have $f(z) = x^2 + y^2$, i.e. $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$. So,
$$u_x = 2x, \qquad u_y = 2y, \qquad v_x = 0 = v_y,$$
and so the Cauchy-Riemann equations are only satisfied at $z = 0$. In particular f is not differentiable for $z \neq 0$. At 0, we have
$$\lim_{h\to 0} \frac{|h|^2}{h} = \lim_{h\to 0} |h| = 0,$$
and so $f'(0) = 0$.

2. We have $u(x, 0) = u(0, y) = 0$ and $v(x, y) = 0$. So, at $(x, y) = (0, 0)$, we have that all the partial derivatives are 0. Hence the Cauchy-Riemann equations are satisfied. However, expressing the function in polar coordinates, we have
$$f(z) = r\sqrt{\cos\theta \sin\theta}.$$
Hence, letting $r \to 0^+$ along the line $\theta = \pi/4$, we find that
$$\lim_{r\to 0^+} \frac{r\sqrt{\cos\theta \sin\theta}}{r(\cos\theta + i\sin\theta)} = \frac{1}{1+i} \neq 0,$$
and so f can not be differentiable at 0.

3. It holds that
$$\Delta\left(ax^3 + bx^2y + cxy^2 + dy^3\right) = 6ax + 2by + 2cx + 6dy.$$
So, for the function to be harmonic, we require $c = -3a$, $b = -3d$.

   Now, observe
$$z^3 = x^3 - 3xy^2 + i(3x^2y - y^3),$$
$$iz^3 = y^3 - 3x^2y + i(x^3 - 3xy^2).$$
Hence
$$\mathrm{Re}\left((a + id)z^3\right) = ax^3 - 3dx^2y - 3axy^2 + dy^3,$$
and so $(a + id)z^3$ is the corresponding analytic function.

   Moreover, the conjugate harmonic function is given by
$$\mathrm{Im}\left((a + id)z^3\right) = a(3x^2y - y^3) + d(x^3 - 3xy^2).$$

4. (a) Using the hint, we see that: for $|z| = 1$,
$$2\sin^2\theta = 1 - \cos(2\theta) = \mathrm{Re}\left(1 - \cos(2\theta) - i\sin(2\theta)\right) = \mathrm{Re}\left(1 - z^2\right).$$
Hence $f(z) = 1 - z^2$ is an analytic function such that $\mathrm{Re}(f)(z)$ agrees with T on the boundary of the unit disc. Since $\mathrm{Re}(f)(z)$ is harmonic, then we must have that
$$T(x, y) = \mathrm{Re}(f)(z) = 1 - x^2 + y^2$$
throughout the disc.

(b) The isothermals are the intersection of the hyperbolae described by $x^2 - y^2 = C$ for $C \in [-1, 1]$ with the unit disc.

(c) The heat flow lines are given by the intersection of the hyperbolae described by $\mathrm{Im}(f)(z) = -2xy = C$ for $C \in [-1, 1]$ with the unit disc.

5. (a) If $f(z) = (1-z)^{-m}$, we have that

$$f^{(n)}(z) = \frac{m \times (m+1) \times \cdots \times (m+n-1)}{(1-z)^{m+n}}.$$

Hence, using the Taylor-Maclaurin expansion,

$$f(z) = \sum_{n=0}^{\infty} \binom{m+n-1}{n} z^n.$$

(b) We have that

$$\frac{1}{1+z} = \frac{1}{2}\left(\frac{1}{1+\frac{z-1}{2}}\right) = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \left(\frac{z-1}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n 2^{-(n+1)}(z-1)^n.$$

Radius of convergence:

$$\left(\limsup_{n\to\infty} \sqrt[n]{2^{-(n+1)}}\right)^{-1} = 2.$$

6. We know that

$$R = \left(\limsup_{n\to\infty} \sqrt[n]{|a_n|}\right)^{-1}.$$

(a) Writing $\sum_n a_n z^{2n} = \sum_{n \text{ even}} a_{n/2} z^n$, we see that the radius of convergence is given by

$$\left(\limsup_{n\to\infty,\, n \text{ even}} \sqrt[n]{|a_{n/2}|}\right)^{-1} = \left(\limsup_{n\to\infty} \sqrt[2n]{|a_n|}\right)^{-1} = R^{1/2}.$$

(b) We have

$$\limsup_{n\to\infty} \sqrt[n]{|a_n^2|} = \limsup_{n\to\infty} \left(\sqrt[n]{|a_n|}\right)^2 = R^{-2},$$

i.e. the radius of convergence of the power series is given by R^2.

7. (a) We can read the real and imaginary parts from:

$$e^{e^z} = e^{e^x(\cos y + i \sin y)} = e^{e^x \cos y}\left(\cos\left(e^x \sin y\right) + i \sin\left(e^x \sin y\right)\right).$$

(b) In this case:

$$\begin{aligned} z^z &= e^{z \log z} \\ &= e^{(x+iy)(\log|z| + i \arg(z))} \\ &= e^{z \log|z| - y \arg(z)}\left(\cos\left(y \log|z| + x \arg(z)\right) + i \sin\left(y \log|z| + x \arg(z)\right)\right). \end{aligned}$$